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CONSTRUCTION OF STATIONARY DISCS FOR PERTURBATIONS OF DECOUPLED SUBMANIFOLDS IN $\mathbb{C}^4$

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Abstract. We construct generalized stationary discs to perturbations of decoupled real submanifolds of codimension 2 in $\mathbb{C}^4$.

Introduction

The method of stationary discs introduced by Lempert ([13], see also [12, 14]) has recently proven to be well adapted in the study of the jet determination of CR diffeomorpshims between real submanifolds of $\mathbb{C}^N$ [1, 2, 3, 5, 15]. One of the key features of this family of discs is the fact that they usually form a submanifold (of the Banach space of analytic discs) of finite dimension. The existence of stationary discs relies on a Riemann-Hilbert type problem and is well understood in many cases; see for instance [1, 2, 6, 14, 15] for nondegenerate real submanifolds, and [3, 5] for degenerate real hypersurfaces. It is then natural to study these discs for more general submanifods and in particular for degenerate real submanifolds of higher codimension. As a first step in this program, we construct stationary discs to pertubations of a decoupled degenerate submanifold of codimension 2 in $\mathbb{C}^4$ (Theorem 2.4). Similarly to [3, 5], we need consider generalized stationary discs in order to take into account the order of degeneracy of the given submanifold.

1. Preliminaries

We denote by Δ the unit disc in $\mathbb{C}$ and by $\partial\Delta$ its boundary. For a positive integer $N > 0$, the set $Gl_N(\mathbb{C})$ denotes the general linear group on $\mathbb{C}^N$.

Let $M \subset \mathbb{C}^4$ be a finitely smooth real submanifold of real codimension 2 given locally by

$$\begin{cases} r_1 = \Re e w_1 - P_1(z_1, \overline{z_1}) + O(d_1 + 1) = 0 \\ r_2 = \Re e w_2 - P_2(z_2, \overline{z_2}) + O(d_2 + 1) = 0 \end{cases} \tag{1.1}$$

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where P_1 and P_2 are real homogenous polynomials, respectively in $z_1, \overline{z_1}$ and $z_2, \overline{z_2}$, with no harmonic part, and of respective degrees d_1 and d_2 with $d_1 \leq d_2$. For $\ell = 1, 2$, we write

$$P_\ell(z_\ell, \overline{z_\ell}) = \sum_{j=d_\ell-k_\ell}^{k_\ell} \alpha_{\ell j} z_\ell^j \overline{z_\ell}^{d_\ell - j}$$

where $d_\ell/2 \leq k_\ell \leq d_\ell - 1$. In the remainder $O(d_\ell + 1)$, $\ell = 1, 2$, z is of weight 1 and $\Im m w$ of weight d_ℓ. We set $r := (r_1, r_2)$ and we write $M = \{r = 0\}$. We associate to M, its model submanifold $M_H = \{\rho = 0\}$ where $\rho := (\rho_1, \rho_2)$ with

$$\begin{cases} \rho_1 = \Re e w_1 - P_1(z_1, \overline{z_1}) = 0 \\ \rho_2 = \Re e w_2 - P_2(z_2, \overline{z_2}) = 0\,. \end{cases} \tag{1.2}$$

An analytic disc f is *attached to M* whenever $f(\partial\Delta) \subset M$. Following [3, 5] (see also Lempert [13] and Tumanov [14]), we define:

Definition 1.1. Let $k_0 > 0$ be a positive integer. A holomorphic disc $f\colon \Delta \to \mathbb{C}^4$ continuous up to $\partial\Delta$ and attached to $M = \{r = 0\}$ is a *k_0-stationary disc for M* if there exists a holomorphic lift $\boldsymbol{f} = (f, \tilde{f})$ of f to the cotangent bundle $T^*\mathbb{C}^4$, continuous up to $\partial\Delta$ and such that for all $\zeta \in \partial\Delta$, $\boldsymbol{f}(\zeta)$ belongs to

$$\mathcal{N}^{k_0}M(\zeta) := \left\{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^4 \mid (z, w) \in M, (\tilde{z}, \tilde{w}) \in \zeta^{k_0} N^*_{(z,w)}M \setminus \{0\}\right\}, \tag{1.3}$$

where

$$N^*_{(z,w)}M = \operatorname{span}_{\mathbb{R}}\{\partial r_1(z, w), \partial r_2(z, w)\}$$

is the conormal fiber at (z, w) of M. The map $\boldsymbol{f} = (f, \tilde{f})$ is called a *k_0-stationary lift for M* and we denote by $\mathcal{S}(M)$ the set of such lifts with f non-constant.

Equivalently, an analytic disc f attached to M is k_0-stationary for M if there are two continuous functions c_1, $c_2\colon \partial\Delta \to \mathbb{R}$ with $\sum_{\ell=1}^2 c_\ell(\zeta)\partial r_\ell(0) \neq 0$ for all $\zeta \in \partial\Delta$ and such that the map

$$\zeta \mapsto \zeta^{k_0} \sum_{\ell=1}^{2} c_\ell(\zeta) \partial r_\ell\left(f(\zeta), \overline{f(\zeta)}\right)$$

defined on $\partial\Delta$ extends holomorphically on Δ. We now provide a basic example.

Example 1.2. Consider a model submanifold $M_H = \{\rho = 0\} \subset \mathbb{C}^4$ of the form (1.2). We have

$$\begin{cases} \partial\rho_1 = (\partial_z\rho_1, \partial_w\rho_1) = \Big(- P_{1,z_1}(z_1, \overline{z}_1), 0, \frac{1}{2}, 0\Big) \\ \partial\rho_2 = (\partial_z\rho_2, \partial_w\rho_2) = \Big(0, -P_{2,z_2}(z_2, \overline{z}_2), 0, \dfrac{1}{2}\Big) \end{cases}$$

where we use the notation $P_{\ell,z_\ell} = \partial_{z_\ell} P_\ell$. We set $k_0 := \max\{k_1, k_2\}$. Then the disc

$$\begin{aligned}\boldsymbol{f_0}(\zeta) &= \big(\underbrace{h_0(\zeta), g_0(\zeta)}_{f_0(\zeta)}, \tilde{h_0}(\zeta), \tilde{g_0}(\zeta)\big)\\ &= \left(1-\zeta, 1-\zeta, g_0(\zeta), \tilde{h_0}(\zeta), \frac{c_1}{2}\zeta^{k_0}, \frac{c_2}{2}\zeta^{k_0}\right), \qquad (1.4)\end{aligned}$$

where c_1, $c_2 \in \mathbb{R}$, not both zero, is a k_0-stationary lift for M_H. Note that g_0 is determined directly by (1.2), while

$$\tilde{h_0}(\zeta) = \zeta^{k_0}\big(c_1 P_{1,z_1}(1-\zeta, 1-\overline{\zeta}), c_2 P_{2,z_2}(1-\zeta, 1-\overline{\zeta})\big)\,.$$

Finally, we introduce the Banach spaces of functions we will work on. For an integer $k \geq 0$ and $0 < \alpha < 1$, we denote by $\mathcal{C}^{k,\alpha}$ the space of real-valued functions defined on $\partial\Delta$ of class $\mathcal{C}^{k,\alpha}$. This space is equipped with its usual norm. We consider $\mathcal{C}^{k,\alpha}_{\mathbb{C}} = \mathcal{C}^{k,\alpha} + i\mathcal{C}^{k,\alpha}$ endowed with the following norm

$$\|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}} = \|\Re e f\|_{\mathcal{C}^{k,\alpha}} + \|\Im m f\|_{\mathcal{C}^{k,\alpha}}\,.$$

The subspace of *analytic discs* $\mathcal{A}^{k,\alpha} \subset \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ consists of functions $f\colon \overline{\Delta} \to \mathbb{C}$ which are holomorphic on Δ and such that $f|_{\partial\Delta} \in \mathcal{C}^{k,\alpha}_{\mathbb{C}}$.

We also introduce spaces with pointwise constraints. Let $m \geq 1$ be an integer. We denote by $\mathcal{C}^{k,\alpha}_{0^m}$ the subspace of functions in $\mathcal{C}^{k,\alpha}$ that can be written as $(1-\zeta)^m v$ with $v \in \mathcal{C}^{k,\alpha}_{\mathbb{C}}$. This space is equipped with the norm $\|(1-\zeta)^m f\|_{\mathcal{C}^{k,\alpha}_{0^m}} = \|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}}$. Finally, we define the subspace $\mathcal{A}^{k,\alpha}_{0^m} \subset \mathcal{C}^{k,\alpha}_{\mathbb{C}}$ of functions of the form $(1-\zeta)^m f$, with $f \in \mathcal{A}^{k,\alpha}$, equipped with the norm $\|(1-\zeta)^m f\|_{\mathcal{A}^{k,\alpha}_{0^m}} = \|f\|_{\mathcal{C}^{k,\alpha}_{\mathbb{C}}}$. When $m = 1$, we simply write $\mathcal{A}^{k,\alpha}_0$ and $\mathcal{C}^{k,\alpha}_0$.

2. CONSTRUCTION OF GENERALIZED STATIONARY DISCS

Consider a decoupled model submanifold M_H given by (1.2). In that case, we obtain explicit defining equations for the fibration $\mathcal{N}^{k_0}M_H(.)$ defined in (1.3). Indeed we have

$$(z, w, \tilde{z}, \tilde{w}) \in \mathcal{N}^{k_0}M_H(\zeta) \Leftrightarrow \begin{cases}\rho_1(z, w, \overline{z}, \overline{w}) = \rho_2(z, w, \overline{z}, \overline{w}) = 0\\ \exists\ c_\ell\colon \partial\Delta \to \mathbb{R}, (\tilde{z}, \tilde{w}) = \zeta^{k_0}\sum_{\ell=1}^2 c_\ell(\zeta)\partial\rho_\ell\,(z, \overline{z})\,.\end{cases}$$

Due to the form of ρ, we have

$$\sum_{\ell=1}^{2} c_\ell(\zeta)\partial\rho_\ell(z, \overline{z}) = \Big(-c_1(\zeta)P_{1,z_1}(z_1, \overline{z_1}), -c_2(\zeta)P_{2,z_2}(z_2, \overline{z_2}), \frac{c_1(\zeta)}{2}, \frac{c_2(\zeta)}{2}\Big)\,.$$

By a straitghtforward computation, it follows that the 8 defining equations of $\mathcal{N}^{k_0}M_H(\zeta)$ are given by

$$\begin{cases}
\tilde\rho_1(\zeta)(z,w,\tilde z,\tilde w) &= \Re e w_1 - P_1(z_1,\overline{z}_1) = 0\\
\tilde\rho_2(\zeta)(z,w,\tilde z,\tilde w) &= \Re e w_2 - P_2(z_2,\overline{z}_2) = 0\\
\tilde\rho_3(\zeta)(z,w,\tilde z,\tilde w) &= (\tilde{z_1} + 2\tilde{w_1}P_{1,z_1}(z_1,\overline{z_1})) + \left(\overline{\tilde{z_1} + 2\tilde{w_1}P_{1,z_1}(z_1,\overline{z_1})}\right) = 0\\
\tilde\rho_4(\zeta)(z,w,\tilde z,\tilde w) &= i\,(\tilde{z_1} + 2\tilde{w_1}P_{1,z_1}(z_1,\overline{z_1})) - i\left(\overline{\tilde{z_1} + 2\tilde{w_1}P_{1,z_1}(z_1,\overline{z_1})}\right) = 0\\
\tilde\rho_5(\zeta)(z,w,\tilde z,\tilde w) &= (\tilde{z_2} + 2\tilde{w_2}P_{2,z_2}(z_2,\overline{z_2})) + \left(\overline{\tilde{z_2} + 2\tilde{w_2}P_{2,z_2}(z_2,\overline{z_2})}\right) = 0\\
\tilde\rho_5(\zeta)(z,w,\tilde z,\tilde w) &= i\,(\tilde{z_2} + 2\tilde{w_2}P_{2,z_2}(z_2,\overline{z_2})) - i\left(\overline{\tilde{z_2} + 2\tilde{w_2}P_{2,z_2}(z_2,\overline{z_2})}\right) = 0\\
\tilde\rho_7(\zeta)(z,w,\tilde z,\tilde w) &= i\frac{\tilde{w_1}}{\zeta^{k_0}} - i\zeta^{k_0}\overline{\tilde{w_1}} = 0\\
\tilde\rho_8(\zeta)(z,w,\tilde z,\tilde w) &= i\frac{\tilde{w_2}}{\zeta^{k_0}} - i\zeta^{k_0}\overline{\tilde{w_2}} = 0\,.
\end{cases}$$

We set $\tilde\rho := (\tilde{\rho_1},\ldots,\tilde{\rho_8})$. For a general submanifold $M = \{r = 0\}$ of the form (1.1), we denote by $\tilde r$ the corresponding defining functions of $\mathcal{N}^{k_0}M(\zeta)$. This allows to consider stationary lifts as solutions of a nonlinear Riemann-Hilbert type problem. More precisely, an analytic disc $\boldsymbol{f}\colon \overline{\Delta} \mapsto T^*\mathbb{C}^4$ is a k_0-stationary lift for M if and only if

$$\tilde r(\boldsymbol{f}) = 0 \quad \text{on } \partial\Delta\,.$$

The study of this problem depends essentially on an appropriate application of the implicit function theorem. Accordingly, we consider the following Banach spaces

$$Y = \left(\mathcal{A}_0^{k,\alpha}\right)^2 \times \left(\mathcal{A}_0^{k,\alpha}\right)^2 \times \mathcal{A}^{k,\alpha}_{0^{d_1-1}} \times \mathcal{A}^{k,\alpha}_{0^{d_2-1}} \times \left(\mathcal{A}^{k,\alpha}\right)^2$$

$$Z = \left(\mathcal{C}_0^{k,\alpha}\right)^2 \times \left(\mathcal{C}^{k,\alpha}_{0^{d_1-1}}\right)^2 \times \left(\mathcal{C}^{k,\alpha}_{0^{d_2-1}}\right)^2 \times \left(\mathcal{C}^{k,\alpha}\right)^2\,.$$

We will also work with a Banach space of admissible defining functions that we will define later, in Subsection 2.2; we will denote this space by X for the time being. We note that although it is important for our approach that the model submanifold is decoupled, we allow for not necessarily decoupled perturbations.

Remark 2.1. The integer k is of little relevance for our work and will not be determined. Essentially, it directly related to the degree d_1, d_2.

We now fix an initial model submanifold M_H (1.2) and an initial stationary lift $\boldsymbol{f_0}$ (1.4), and we define the map $F : X \times Y \to Z$ in a neighborhood of $(\rho, \boldsymbol{f_0})$ in $X \times Y$ by

$$F(r,\boldsymbol{f}) := \tilde r(\boldsymbol{f})\,. \tag{2.1}$$

Here, we use the notation $\tilde r(\boldsymbol{f})(\zeta) = \tilde r(\zeta)(\boldsymbol{f}(\zeta))$ for $\zeta \in \partial\Delta$. The map F is of class $\mathcal{C}^1$ (see Lemma 5.1 in [11] and Lemma 6.1 and Lemma 11.2 in [9]). And the zero set of $F(r,\cdot)$ coincides with the set $\mathcal{S}(\{r = 0\})$ of stationary lifts for $\{r = 0\}$. In order to apply the implicit function theorem to F, we consider the partial derivative of the map F with respect to the Banach space Y at $(\rho, \boldsymbol{f_0})$, that is,

$$\boldsymbol{f} \mapsto \partial_2 F(\rho, \boldsymbol{f_0})\boldsymbol{f} = 2\Re e\left[\overline{G(\zeta)}\boldsymbol{f}\right]$$

where $G(\zeta)$ is the following complex valued 8×8 matrix

$$G(\zeta) := \left(\tilde\rho_{\overline{w}}(\boldsymbol{f_0}), \tilde\rho_{\overline{z}}(\boldsymbol{f_0}), \tilde\rho_{\overline{\tilde z}}(\boldsymbol{f_0}), \tilde\rho_{\overline{\tilde w}}(\boldsymbol{f_0})\right). \tag{2.2}$$

We point out that it is more convenient to reorder coordinates and work with $(w, z, \tilde z, \tilde w)$ instead of $(z, w, \tilde z, \tilde w)$; so, discs $\boldsymbol{f}$ are of the form $(g, h, \tilde h, \tilde g)$. In order to construct stationary lifts near $\boldsymbol{f_0}$ attached to small perturbations of $\{\rho = 0\}$, we then need to

i. show that the map $\partial_2 F(\rho, \boldsymbol{f_0})\colon Y \to Z$ is onto, and

ii. determine the real dimension of its kernel (see p. 39 [10]).

This relies entirely on the values of particular integers, namely the partial indices and the Maslov index associated to the matrix G [8, 9, 10]. We briefly recall these notions. Let $A\colon \partial\Delta \to Gl_N(\mathbb{C})$ be a smooth map. We consider a Birkhoff factorization (see Section 3 [9] or [16]) of $-\overline{A^{-1}}A$ on $\partial\Delta$:

$$-\overline{A(\zeta)}^{-1}A(\zeta) = B^+(\zeta)\begin{pmatrix} \zeta^{\kappa_1} & & & (0) \\ & \zeta^{\kappa_2} & & \\ & & \ddots & \\ (0) & & & \zeta^{\kappa_N} \end{pmatrix} B^-(\zeta),$$

where $\zeta \in \partial\Delta$, $B^+\colon \bar\Delta \to Gl_N(\mathbb{C})$ and $B^-\colon (\mathbb{C} \cup \infty) \setminus \Delta \to Gl_N(\mathbb{C})$ are smooth maps, holomorphic on Δ and $\mathbb{C} \setminus \overline{\Delta}$ respectively. The integers $\kappa_1, \ldots, \kappa_N$ are the *partial indices* of $-\overline{A^{-1}}A$ and their sum $\kappa := \sum_{j=1}^N \kappa_j$ is the *Maslov index* of $-\overline{A^{-1}}A$. In the next section, we illustrate this approach on a toy decoupled model.

2.1. **A toy example.** We consider the model submanifold M_H given by

$$\begin{cases} \rho_1 = \Re e w_1 - |z_1|^4 = 0 \\ \rho_2 = \Re e w_2 - |z_2|^6 = 0 \end{cases}$$

In that case, $k_0 = 3$ and the defining equations of $\mathcal{N}^3 M_H(\zeta)$ are given by

$$\begin{cases} \tilde\rho_1(\zeta)(z, w, \tilde z, \tilde w) & = & \Re e w_1 - |z_1|^4 = 0 \\ \tilde\rho_2(\zeta)(z, w, \tilde z, \tilde w) & = & \Re e w_2 - |z_2|^6 = 0 \\ \tilde\rho_3(\zeta)(z, w, \tilde z, \tilde w) & = & \left(\tilde z_1 + 4\tilde w_1 z_1 \overline{z_1^2}\right) + \left(\overline{\tilde z_1 + 4\tilde w_1 z_1 \overline{z_1^2}}\right) = 0 \\ \tilde\rho_4(\zeta)(z, w, \tilde z, \tilde w) & = & i\left(\tilde z_1 + 4\tilde w_1 z_1 \overline{z_1^2}\right) - i\left(\overline{\tilde z_1 + 4\tilde w_1 z_1 \overline{z_1^2}}\right) = 0 \\ \tilde\rho_5(\zeta)(z, w, \tilde z, \tilde w) & = & \left(\tilde z_2 + 6\tilde w_2 z_2^2 \overline{z_2^3}\right) + \left(\overline{\tilde z_2 + 6\tilde w_2 z_2^2 \overline{z_2^3}}\right) = 0 \\ \tilde\rho_6(\zeta)(z, w, \tilde z, \tilde w) & = & i\left(\tilde z_2 + 6\tilde w_2 z_2^2 \overline{z_2^3}\right) - i\left(\overline{\tilde z_2 + 6\tilde w_2 z_2^2 \overline{z_2^3}}\right) = 0 \\ \tilde\rho_7(\zeta)(z, w, \tilde z, \tilde w) & = & i\dfrac{\tilde w_1}{\zeta^3} - i\zeta^3\overline{\tilde w_1} = 0 \\ \tilde\rho_8(\zeta)(z, w, \tilde z, \tilde w) & = & i\dfrac{\tilde w_2}{\zeta^3} - i\zeta^3\overline{\tilde w_2} = 0\,. \end{cases}$$

We consider the initial 3-stationary lift for M_H

$$\boldsymbol{f_0}(\zeta) = (h_0(\zeta), g_0(\zeta), \tilde{h_0}(\zeta), \tilde{g_0}(\zeta)) = \left(1 - \zeta, 1 - \zeta, g_0(\zeta), \tilde{h_0}(\zeta), \frac{\zeta^3}{4}, \frac{\zeta^3}{6}\right).$$

The matrix map $G(\zeta)$ defined in (2.2) is

$$G(\zeta) = \begin{pmatrix} \frac{1}{2}I_2 & & (*) \\ & G_2(\zeta) & \\ (0) & & -i\zeta^3 I_2 \end{pmatrix}, \tag{2.3}$$

with

$$G_2 = \begin{pmatrix} 2\zeta^3|1-\zeta|^2 + \bar{\zeta}^3(1-\zeta)^2 & 0 & 1 & 0 \\ 2i\zeta^3|1-\zeta|^2 - i\bar{\zeta}^3(1-\zeta)^2 & 0 & -i & 0 \\ 0 & 3\zeta^3|1-\zeta|^4 + 2\bar{\zeta}^3|1-\zeta|^2(1-\zeta)^2 & 0 & 1 \\ 0 & 3i\zeta^3|1-\zeta|^4 - 2i\bar{\zeta}^3|1-\zeta|^2(1-\zeta)^2 & 0 & -i \end{pmatrix}.$$

We emphasize that, due to the form of G (2.3), the surjectivity of $\partial_2 F(\rho, \boldsymbol{f_0})$ amounts to the surjectivity of the linear map

$$L_2 : \left(\mathcal{A}_0^{k,\alpha}\right)^2 \times \mathcal{A}_{0^3}^{k,\alpha} \times \mathcal{A}_{0^5}^{k,\alpha} \to \left(\mathcal{C}_{0^3}^{k,\alpha}\right)^2 \times \left(\mathcal{C}_{0^5}^{k,\alpha}\right)^2.$$

defined by

$$L_2((1-\zeta)h, (1-\zeta)^3\tilde{h_1}, (1-\zeta)^5\tilde{h_2}) = 2\Re e\left[\overline{G_2(\zeta)}((1-\zeta)h, (1-\zeta)^3\tilde{h_1}, (1-\zeta)^5\tilde{h_2})\right].$$

A direct computation gives

$$G_2(\zeta) = \begin{pmatrix} (1-\overline{\zeta})^2(-2\zeta^4 + \overline{\zeta}) & 0 & 1 & 0 \\ -i(1-\overline{\zeta})^2(2\zeta^4 + \overline{\zeta}) & 0 & -i & 0 \\ 0 & (1-\overline{\zeta})^4(3\zeta^5 - 2) & 0 & 1 \\ 0 & i(1-\overline{\zeta})^4(3\zeta^5 + 2) & 0 & -i \end{pmatrix}.$$

We point out that the matrices $G_2(\zeta)$ and thus $G(\zeta)$ are not invertible at $\zeta = 1$. We will desingularize these matrices by decomposing them in order make use of the partial index method. We first permute the second and third columns

$$G_2(\zeta) = \begin{pmatrix} (1-\overline{\zeta})^2(-2\zeta^4 + \overline{\zeta}) & 1 & 0 & 0 \\ -2i(1-\overline{\zeta})^2(2\zeta^4 + \overline{\zeta}) & -i & 0 & 0 \\ 0 & 0 & 3(1-\overline{\zeta})^4(3\zeta^5 - 2) & 1 \\ 0 & 0 & 3i(1-\overline{\zeta})^4(3\zeta^5 + 2) & -i \end{pmatrix},$$

and factorize

$$G_2(\zeta) = \underbrace{\begin{pmatrix} -2\zeta^4 + \overline{\zeta} & 1 & 0 & 0 \\ -2i\zeta^4 - i\overline{\zeta} & -i & 0 & 0 \\ 0 & 0 & 3\zeta^5 - 2 & 1 \\ 0 & 0 & 3i\zeta^5 + 2i & -i \end{pmatrix}}_{\widetilde{G_2}(\zeta)\ \in\ Gl_4(\mathbb{C})} \times \underbrace{\begin{pmatrix} (1-\overline{\zeta})^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-\overline{\zeta})^4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{D(\zeta)}.$$

It turns out that the linear operator

$$\widetilde{L_2} \colon \left(\mathcal{A}_{0^3}^{k,\alpha}\right)^2 \times \left(\mathcal{A}_{0^5}^{k,\alpha}\right)^2 \to \left(\mathcal{C}_{0^3}^{k,\alpha}\right)^2 \times \left(\mathcal{C}_{0^5}^{k,\alpha}\right)^2$$

defined by

$$L_2 = \widetilde{L_2} \circ \overline{D}$$

is exactly of the form considered in Theorem 2.4 in [4], with $m_1 = 3$ and $m_2 = 5$, and its surjectivity is equivalent to the one of L_2. In order to show its surjectivity, we then need to study the matrix $-\overline{\widetilde{G_2}^{-1}}\widetilde{G_2}$ and show that its partial indices $\kappa_1, \ldots, \kappa_4$ are such that $k_1, k_2 \geq m_1 - 1 = 2$ and $k_3, k_4 \geq m_2 - 1 = 4$. We first write

$$\widetilde{G_2}(\zeta) = \begin{pmatrix} B_1 & (0) \\ (0) & B_2 \end{pmatrix}$$

and note that we have $|B_1| = 4i\zeta^4$, $|B_2| = -6i\zeta^5$, and thus $\left|-\overline{\widetilde{G_2}^{-1}}\widetilde{G_2}\right| = \zeta^{18}$. Moreover,

$$-\overline{B_1^{-1}} = \begin{pmatrix} \frac{\zeta^4}{4} & i\frac{\zeta^4}{4} \\ -\frac{2+\zeta^5}{4} & i\frac{2-\zeta^5}{4} \end{pmatrix} \text{ and } -\overline{B_2^{-1}} = \begin{pmatrix} -\frac{\zeta^5}{6} & -i\frac{\zeta^5}{6} \\ -\frac{3+2\zeta^5}{6} & i\frac{3-2\zeta^5}{6} \end{pmatrix}$$

from which it follows that

$$-\overline{\widetilde{G_2}^{-1}}\widetilde{G_2} = \begin{pmatrix} \frac{\zeta^3}{2} & \frac{\zeta^4}{2} & 0 & 0 \\ \frac{3\zeta^4}{2} & -\frac{\zeta^5}{2} & 0 & 0 \\ 0 & 0 & \frac{2\zeta^5}{3} & -\frac{\zeta^5}{3} \\ 0 & 0 & -\frac{5\zeta^5}{3} & \frac{-2\zeta^5}{3} \end{pmatrix}$$

that can be factorized as

(2.4)
$$\begin{pmatrix} 2+\zeta & 1 & 0 & 0 \\ -2i+i\zeta & i & 0 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & i & i \end{pmatrix}^{-1} \begin{pmatrix} \zeta^4 & 0 & 0 & 0 \\ 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & \zeta^5 & 0 \\ 0 & 0 & 0 & \zeta^5 \end{pmatrix} \begin{pmatrix} 2+\overline{\zeta} & 1 & 0 & 0 \\ 2i-i\overline{\zeta} & -i & 0 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & -i & -i \end{pmatrix}.$$

Thus, the partial indices are $\kappa_1 = \kappa_2 = 4$ and $\kappa_3 = \kappa_4 = 5$, and the maps $\widetilde{L_2}$, L_2 and $\partial_2 F(\rho, \boldsymbol{f_0})$ are onto.

We now focus on the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ and show

Lemma 2.2. *The real dimension of the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ is* 20.

Proof. The proof relies once more on Theorem 2.4 [4] applied to matrix G (2.3). Using the same notation, we have $r = 4$, $m_1 = 1$, $m_2 = 3$, $m_3 = 5$, $m_4 = 0$, and $N_1 = N_2 = N_3 = N_4 = 2$. The Maslov index κ is given the sum of the partial indices $0, 0, 4, 4, 5, 5, 6, 6$ and is then $\kappa = 30$. Thus, the dimension of the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ is equal to

$$\kappa + 8 - \sum_{j=1}^{4} N_j m_j = 38 - 2(1+3+5+0) = 20\,.$$

□

Finally, this shows that, for any defining function $r \in X$ close enough to ρ, the set of stationary lifts for $\{r = 0\}$ near $\boldsymbol{f_0}$ is a $\mathcal{C}^1$ submanifold of the Banach space Y of finite real dimension 20.

Remark 2.3. Due to the nature of this example, we cared about finding the explicit factorization (2.4). This can be in general avoided. Indeed, such a factorization relies on a system of linear equations whose solvability imposes conditions on the partial indices. We will use this approach in the general case exposed in the next section.

2.2. **Our main result.** We first discuss the space of defining functions X we will be working on. Choose $\delta > 0$ large enough so that $f_0\left(\overline{\Delta}\right)$, where f_0 is defined in (1.4), is contained in the polydisc $\delta\Delta^4 \subset \mathbb{C}^4$. Following [3, 5], we consider the affine Banach space X of functions $r \in \mathcal{C}^{k+3}\left(\delta\Delta^4\right)$ which can be written as

$$r(z,w) = \rho(z,w) + \theta(z, \Im mw)$$

with $\theta = (\theta_1, \theta_2)$ of the form

$$\theta_\ell(z, \Im mw) = \sum_{I+J=D_\ell+1} z^I \overline{z}^J r_{\ell,IJ0}(z) + \sum_{l=1}^{D_\ell} \sum_{I+J=D_\ell - l} z^I \overline{z}^J (\Im mw)^l r_{\ell,IJl}(z, \Im mw)$$

where $r_{\ell,IJ0} \in \mathcal{C}^{k+3}_{\mathbb{C}}\left(\delta\Delta^2\right)$ and $r_{\ell,IJl} \in \mathcal{C}^{k+3}_{\mathbb{C}}\left(\delta\Delta^2 \times (-\delta,\delta)^2\right)$. The space X is equipped with the norm

$$\|r\|_X = \sup \|r_{\ell,IJl}\|_{\mathcal{C}^{k+3}},$$

and is then a Banach space since it is isomorphic to a real closed subspace of (a suitable power of) $\mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^2} \times (-\delta,\delta)^2\right)$.

We now state our main result

Theorem 2.4. *Let* $M_H = \{\rho = 0\} \subset \mathbb{C}^4$ *be a decoupled model submanifold of the form* (1.2). *We assume that the zero sets of the Laplacians of* P_1 *and* P_2 *are respectively* $\{0\} \times \mathbb{C}$ *and* $\mathbb{C} \times \{0\}$. *Consider an initial lift of a stationary disc* $\boldsymbol{f_0} = (h_0, g_0, \tilde{h_0}, \tilde{g_0}) \in Y$ *of the form* (1.4). *Then there exist an open neighborhood* U *of* ρ *in* X *and a real number* $\varepsilon > 0$ *such that for any defining function* $r \in U$, *the set*

$$\{\boldsymbol{f} \in \mathcal{S}(\{r=0\}) \mid \|\boldsymbol{f} - \boldsymbol{f_0}\|_{1,\alpha} < \varepsilon\}$$

forms a $\mathcal{C}^1$ *real submanifold of finite dimension of the Banach space of analytic discs.*

Remark 2.5. The main issue faced in the degenerate setting is the fact that the conormal bundle of M_H is no longer totally real. While the initial disc $\boldsymbol{f_0}$ passes through the singularity of the conormal bundle of M_H at $\zeta = 1$, it is important to control its intersection with that singularity. This is achieved by imposing the vanishing condition of the Laplacians of P_1 and P_2 that allows the reduction to a totally real setting. In practical terms, this corresponds to the invertibility of the matrix map $\widetilde{G_2}(.)$ - in the below proof - that arises from factorizing the noninvertible matrix map $G_2(.)$.

Proof. Following exactly the scheme used in the toy example (Subsection 2.1), we have

$$G(\zeta) = \begin{pmatrix} \frac{1}{2}I_2 & & (*) \\ & G_2(\zeta) & \\ (0) & & -i\zeta^{k_0}I_2 \end{pmatrix},$$

where G_2, after permutation of its second and third columns, is given by

$$G_2(\zeta) = \begin{pmatrix} \zeta^{k_0}P_{1,z_1\bar{z_1}} + \bar{\zeta}^{k_0}P_{1,\bar{z_1}\bar{z_1}} & 1 & 0 & 0 \\ i(\zeta^{k_0}P_{1,z_1\bar{z_1}} - \bar{\zeta}^{k_0}P_{1,\bar{z_1}\bar{z_1}}) & -i & 0 & 0 \\ 0 & 0 & \zeta^{k_0}P_{2,z_2\bar{z_2}} + \bar{\zeta}^{k_0}P_{2,\bar{z_2}\bar{z_2}} & 1 \\ 0 & 0 & i(\zeta^{k_0}P_{2,z_2\bar{z_2}} - \bar{\zeta}^{k_0}P_{2,\bar{z_2}\bar{z_2}}) & -i \end{pmatrix}.$$

We now write

$$\begin{cases} \zeta^{k_0}P_{\ell,z_\ell\bar{z}_\ell}\left(1-\zeta,\overline{1-\zeta}\right) = (1-\overline{\zeta})^{d_\ell-2}Q_\ell(\zeta) \\ \\ \zeta^{k_0}P_{\ell,z_\ell z_\ell}\left(1-\zeta,\overline{1-\zeta}\right) = (1-\overline{\zeta})^{d_\ell-2}S_\ell(\zeta) \end{cases}$$

where Q_ℓ and S_ℓ are holomorphic polynomials. Note that each Q_ℓ has degree at most $k_0+k_\ell-1$ and is divisible by $\zeta^{k_0-k_\ell+d_\ell-1}$, while each S_ℓ has degree at most $k_0+k_\ell-2$ and is divisible by $\zeta^{k_0-k_\ell+d_\ell-2}$. So we have

$$G_2(\zeta) =$$
$$\begin{pmatrix} (1-\overline{\zeta})^{d_1-2}\left(Q_1(\zeta)+\zeta^{d_1-2}\overline{S}_1(\zeta)\right) & 1 & 0 & 0 \\ i(1-\overline{\zeta})^{d_1-2}\left(Q_1(\zeta)-\zeta^{d_1-2}\overline{S}_1\right) & -i & 0 & 0 \\ 0 & 0 & (1-\overline{\zeta})^{d_2-2}\left(Q_2(\zeta)+\zeta^{d_2-2}\overline{S}_2(\zeta)\right) & 1 \\ 0 & 0 & i((1-\overline{\zeta})^{d_2-2}\left(Q_2(\zeta)-\zeta^{d_2-2}\overline{S}_2(\zeta)\right) & -i \end{pmatrix}.$$

and factorize

$$G_2(\zeta) = \underbrace{\begin{pmatrix} Q_1(\zeta)+\zeta^{d_1-2}\overline{S}_1(\zeta) & 1 & 0 & 0 \\ iQ_1(\zeta)-i\zeta^{d_1-2}\overline{S}_1(\zeta) & -i & 0 & 0 \\ 0 & 0 & Q_2(\zeta)+\zeta^{d_2-2}\overline{S}_2(\zeta) & 1 \\ 0 & 0 & iQ_2(\zeta)-i\zeta^{d_2-2}\overline{S}_2(\zeta) & -i \end{pmatrix}}_{\widetilde{G_2}(\zeta)}$$
$$\times \underbrace{\begin{pmatrix} (1-\overline{\zeta})^{d_1-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-\overline{\zeta})^{d_2-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{D(\zeta)}.$$

It follows that $\widetilde{G_2}(\zeta) \in Gl_4(\mathbb{C})$ since its determinant is equal to $-4Q_1(\zeta)Q_2(\zeta)$ and is non vanishing on $\partial\Delta$ due to our assumption on the zero sets of the Laplacians of P_1 and P_2. Accordingly, the linear operator

$$\widetilde{L_2}\colon \left(\mathcal{A}^{k,\alpha}_{0^{d_1-1}}\right)^2 \times \left(\mathcal{A}^{k,\alpha}_{0^{d_2-1}}\right)^2 \to \left(\mathcal{C}^{k,\alpha}_{0^{d_1-1}}\right)^2 \times \left(\mathcal{C}^{k,\alpha}_{0^{d_2-1}}\right)^2$$

given by $L_2 = \widetilde{L_2} \circ \overline{D}$ is of the form considered in Theorem 2.4 in [4], with $m_1 = d_1 - 1$ and $m_2 = d_2 - 1$. To prove its surjectivity, we will now study the partial indices $\kappa_1, \dots, \kappa_4$ of $-\overline{\widetilde{G_2}^{-1}}\widetilde{G_2}$ and show that they satisfy $k_1, k_2 \geq d_1 - 2$ and $k_3, k_4 \geq d_2 - 2$.

By a direct computation, we obtain

$$-\overline{\widetilde{G_2}^{-1}}\widetilde{G_2} = \begin{pmatrix} D_1 & (0) \\ (0) & D_2 \end{pmatrix} \text{ with } D_\ell = -\frac{1}{\overline{Q}} \begin{pmatrix} \zeta^{d_\ell - 2}\overline{S}_\ell & 1 \\ |Q_\ell|^2 - |S_\ell|^2 & -\overline{\zeta}^{d_\ell - 2} S_\ell \end{pmatrix}.$$

Then there exists a smooth map $\Theta : \overline{\Delta} \to GL_4(\mathbb{C})$, holomorphic on Δ, such that

$$\Theta = \begin{pmatrix} \Theta_1 & (0) \\ (0) & \Theta_2 \end{pmatrix} \text{ and } -\Theta\overline{G_2^{-1}}G_2 = \Lambda\overline{\Theta} \text{ on } \partial\Delta$$

where Λ is the diagonal 4×4 matrix with entries $\kappa_1, \dots, \kappa_4$ (see Lemma 5.1 [9]). We write

$$\Theta_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

and obtain the following system

$$\begin{cases} a_1\zeta^{d_1-2}\overline{S}_1 + b_1(|Q_1|^2 - |S_1|^2) & = & -\overline{Q}_1\zeta^{\kappa_1}\overline{a}_1 \\ a_1 - b_1\overline{\zeta}^{d_1-2}S_1 & = & -\overline{Q}_1\zeta^{\kappa_1}\overline{b}_1 \\ c_1\zeta^{d_1-2}\overline{S}_1 + d_1(|Q_1|^2 - |S_1|^2) & = & -\overline{Q}_1\zeta^{\kappa_2}\overline{c}_1 \\ c_1 - d_1\overline{\zeta}^{d_1-2}S_1 & = & -\overline{Q}_1\zeta^{\kappa_2}\overline{d}_1 \,. \end{cases}$$

Since S_1 is divisible by $\zeta^{k_0 - k_1 + d_1 - 2}$, the left-hand sides, and thus the right-hand sides, of the second and fourth equations are holomorphic. It follows that

$$\kappa_1, \kappa_2 \geq k_0 + k_1 - 2 \geq 2k_1 - 2 \geq d_1 - 2\,.$$

Similarly we obtain

$$\kappa_3, \kappa_4 \geq k_0 + k_2 - 2 \geq 2k_2 - 2 \geq d_2 - 2\,.$$

Theorem 2.4 in [4] then implies that the operator $\widetilde{L_2}$, and thus L_2 and $\partial_2 F(\rho, \boldsymbol{f_0})$, are onto.

We will now focus on the real dimension of the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ which relies once more on Theorem 2.4 [4]. We first write

$$G(\zeta) = \underbrace{\begin{pmatrix} \frac{1}{2}I_2 & & (*) \\ & G_2(\zeta) & \\ (0) & & -i\zeta^{k_0}I_2 \end{pmatrix}}_{\tilde{G}(\zeta)} \begin{pmatrix} I_2 & & (0) \\ & D(\zeta) & \\ (0) & & I_2 \end{pmatrix}$$

and we note that the kernels of $\partial_2 F(\rho, \boldsymbol{f_0})$ and the operator and

$$\left(\mathcal{A}_0^{k,\alpha}\right)^2 \times \left(\mathcal{A}_{0^{d_1-1}}^{k,\alpha}\right)^2 \times \left(\mathcal{A}_{0^{d_2-1}}^{k,\alpha}\right)^2 \times \left(\mathcal{A}^{k,\alpha}\right)^2 \ni \boldsymbol{f} \mapsto 2\Re e\left[\overline{\tilde{G}(\zeta)}\boldsymbol{f}\right]$$

are of the same dimension. Using the same notation as Theorem 2.4 [4], we have $r = 4$, $m_1 = 1$, $m_2 = d_1 - 1$, $m_3 = d_2 - 2$, $m_4 = 0$, and $N_1 = N_2 = N_3 = N_4 = 2$. This time, since we have not determined the partial indices, we will use the fact that the Maslov index κ is also equal to the winding number at the origin of the map

$$\zeta \mapsto \det\left(-\overline{\tilde{G}(\zeta)^{-1}}\tilde{G}(\zeta)\right).$$

See e.g. [10] or Lemma B.1 [6] for a proof of this fact. We obtain directly

$$\kappa = \operatorname{ind}(-\overline{Q_1}^{-1}Q_1) + \operatorname{ind}(-\overline{Q_2}^{-1}Q_2) + 4k_0$$

leading to the dimension of the kernel of $\partial_2 F(\rho, \boldsymbol{f_0})$ to be equal to

$$\kappa + 8 - \sum_{j=1}^{4} N_j m_j = \operatorname{ind}(-\overline{Q_1}^{-1}Q_1) + \operatorname{ind}(-\overline{Q_2}^{-1}Q_2) + 4k_0 + 10 - 2(d_1 + d_2).$$

This achieves the proof of Theorem 2.4. □

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